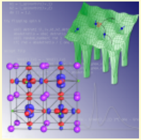
Introduction to Mean-Field Theory of Spin Glass Models

Václav Janiš

Institute of Physics, Academy of Sciences of the Czech Republic, Praha, CZ

Autumn School on Correlated Electrons:
Many-Body Physics: From Kondo to Hubbard
21-25 September 2015, FZ Jülich



Collaborators: Anna Kauch, Antonín Klíč (FZÚ AV ČR, v. v. i.)

Motivation

Why mean-field theory of spin glasses?

- Diluted magnetic impurities (Mn, Fe) in a metallic matrix (Cu. Au, Ag, Pt)
- Fluctuating long-range spin exchange (RKKY) – mean field approximation should be good
- Standard approach fails - inconsistent
- Analytic approach – replica trick & replica-symmetry breaking (no direct physical interpretation)
- New type of long range order: non-measurable order parameters

What is the physical meaning of replica-symmetry breaking?
Can we avoid replicas?

Fundamental concept to follow: Ergodicity

Motivation

Why mean-field theory of spin glasses?

- Diluted magnetic impurities (Mn, Fe) in a metallic matrix (Cu. Au, Ag, Pt)
- Fluctuating long-range spin exchange (RKKY) – mean field approximation should be good
- Standard approach fails - inconsistent
- Analytic approach – replica trick & replica-symmetry breaking (no direct physical interpretation)
- New type of long range order: non-measurable order parameters

What is the physical meaning of replica-symmetry breaking?
Can we avoid replicas?

Fundamental concept to follow: Ergodicity

Motivation

Why mean-field theory of spin glasses?

- Diluted magnetic impurities (Mn, Fe) in a metallic matrix (Cu. Au, Ag, Pt)
- Fluctuating long-range spin exchange (RKKY) – mean field approximation should be good
- Standard approach fails - inconsistent
- Analytic approach – replica trick & replica-symmetry breaking (no direct physical interpretation)
- New type of long range order: non-measurable order parameters

What is the physical meaning of replica-symmetry breaking?
Can we avoid replicas?

Fundamental concept to follow: Ergodicity

Outline

1. Introduction – spin models and mean-field solution
   - Models of interacting spins
   - Models with disorder and frustration – spin glasses
2. Fundamental concepts: Ergodicity, thermodynamic homogeneity and real replicas
   - Ergodicity in statistical physics
   - Real-replica method for restoring thermodynamic homogeneity
3. Hierarchical construction of mean-field theory of spin glasses
   - Discrete replica-symmetry (replica-independence) breaking
   - Continuous replica-symmetry breaking
4. Solvable cases: 1RSB and asymptotic $T \nearrow T_c$ solutions
   - One-level RSB – Ising
   - Infinite RSB - asymptotic solution – Ising
   - Potts and p-spin glass
5. Conclusions

Heisenberg spins

- Model of interacting spins (Heisenberg)

$$H[J,\mathbf{S}] = -\sum_{i<j} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j$$

- Spin exchange
  - Ferromagnetic interaction: $J_{ij} > 0$
  - Antiferromagnetic interaction: $J_{ij} < 0$
- Regular crystalline structure (lattice)
- Strong anisotropy: Only single spin projection (S^z) interacts
- Ising model

$$H[J,S] = -\sum_{i<j} J_{ij}S_iS_j$$

- Classical spins with $S_i = \pm 1$ ($\hbar/2$ units)

Other spin models – generalizations of Ising I

- Potts model - $p > 2$ spin projections

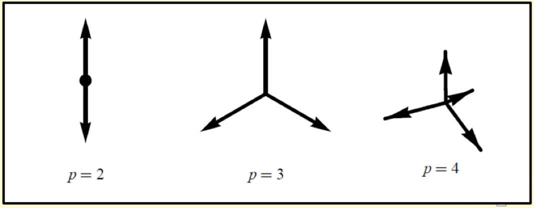
$$H_p = -\sum_{i<j} J_{ij}\delta_{n_i,n_j}$$

$n_i = 1, 2, \ldots, p$

- Spin representation

$$H_P[J, \mathbf{S}] = -\frac{1}{2}\sum_{i,j} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j - \sum_i \mathbf{h} \cdot \mathbf{S}_i\ ,$$

Potts vectors $\mathbf{S}_i = \{s_i^1, \ldots s_i^{p-1}\}$, values are state vectors $\{\mathbf{e}_A\}_{A=1}^p$



Other spin models – generalizations of Ising II

$$\sum_{A=1}^{p} e_A^\alpha = 0\,, \quad \sum_{A=1}^{p} e_A^\alpha e_A^\beta = p\,\delta^{\alpha\beta}\,, \quad \sum_{\alpha=1}^{p-1} e_A^\alpha e_B^\alpha = p\,\delta_{AB} - 1$$

- Explicit representation

$$e_A^\alpha = \begin{cases} 0 & A < \alpha \\ \sqrt{\frac{p(p-\alpha)}{p+1-\alpha}} & A = \alpha \\ \frac{1}{\alpha-p}\sqrt{\frac{p(p-\alpha)}{p+1-\alpha}} & A > \alpha\,. \end{cases}$$

- p-spin model

$$H_p[J,S] = \sum_{1\le i_1<i_2<\ldots<i_p} J_{i_1 i_2\ldots i_p} S_{i_1} S_{i_2}\ldots S_{i_p}\,.$$

S are Ising spins, $p = 2$ reduces to Ising

Ising thermodynamics – mean-field solution

- Thermally induced spin fluctuations – free energy

$$-\beta F(T) = \ln \mathrm{Tr}_S \exp\{-\beta H[J, S]\}$$

- Long-range ferromagnetic interaction: $J_{ij} = -J/N$
- Mean-field (Weiss) solution with ergodic assumption

$$f(T, m) = F(T, m)/N = \frac{Jm^2}{2} - \frac{1}{\beta} \ln 2 \cosh(\beta J m)$$

global magnetization m – variational parameter

- Equilibrium state – magnetization minimizing $f(T, m)$
- Equilibrium magnetization

$$m = \tanh(\beta J m)$$

minimizes free energy

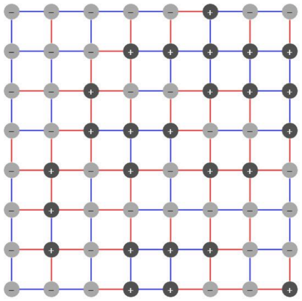
Critical point – ergodicity & symmetry breaking

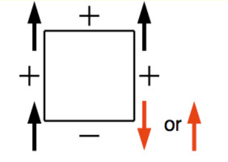
- Critical point $\beta J = 1$ separates two phases
  - Paramagnetic: $m = 0$
  - Ferromagnetic: $1 \geq m^2 > 0$
- Ergodicity broken in the FM phase (in a trivial way)
- Spin-reflection symmetry $H[J, S] = H[J, -S]$ broken
- Non-ergodic situation: degenerate solution ($F(T, m) = F(T, -m)$)
- Adding magnetic energy $H'[h, S] = -h \sum_i S_i$ lifts degeneracy & restores ergodicity

Ergodicity (uniqueness of equilibrium state) restored by a symmetry-breaking magnetic field

Disorder & frustration - inhomogeneous spin exchange I

- Randomness in the spin exchange
- System locally frustrated: ferro (red bond) and antiferro (blue bond) randomly distributed





- Unbiased situation – neither ferro nor antiferro ordering preferred
Gaussian random variables in mean-field limit

Disorder & frustration – inhomogeneous spin exchange II

$$N\left\langle J_{ij}\right\rangle_{av} = \sum_{j=1}^{N} J_{ij} = 0, \quad N\left\langle J_{ij}^2\right\rangle_{av} = \sum_{j=1}^{N} J_{ij}^2 = J^2$$

- Ising model

$$P(J_{ij}) = \sqrt{\frac{N}{2\pi J^2}}\, \exp\left\{-\frac{N J_{ij}^2}{2J^2}\right\}$$

- Potts model (not symmetric v.r.t. spin reflection)

$$P(J_{ij}) = \sqrt{\frac{N}{2\pi J^2}}\, \exp\frac{-N(J_{ij} - J_0/N)^2}{2J^2}\,,$$

$J_0 = \sum_j J_{0j}$ – averaged (ferromagnetic) interaction

Disorder & frustration - inhomogeneous spin exchange III

- p-spin model

$$P\left(J_{i_1 i_2 \ldots i_p}\right) = \sqrt{\frac{N^{p-1}}{\pi p!}} \exp\left\{-\frac{J_{i_1 i_2 \ldots i_p}{}^2 N^{p-1}}{J^2 p!}\right\}$$

Real spin-glass systems

- Highly diluted magnetic ions (Fe, Mn) in noble metals (Au, Cu)
- RKKY interaction – generates effectively random long-range spin exchange
- Critical behavior in magnetic field – FC & ZFC

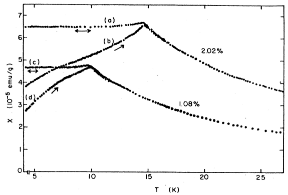


FIG. 7. Static susceptibilities of *Cu*Mn vs temperature for 1.08 and 2.02 at. % Mn. After zero-field cooling ($H < 0.05$ Oe), initial susceptibilities (b) and (d) were taken for increasing temperature in a field of $H = 5.9$ Oe. The susceptibilities (a) and (c) were obtained in the field $H = 5.9$ Oe, which was applied above T_f before cooling the samples. From Nagata *et al.* (1979).

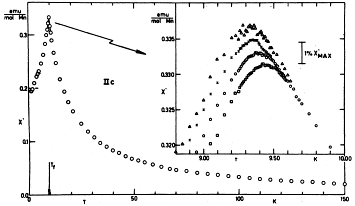


FIG. 1. Real part χ' of the complex susceptibility $\chi(\omega)$ as a function of temperature for sample IIc (*Cu*Mn with 0.94 at. % Mn, powder). Inset reveals frequency dependence and rounding of the cusp by use of strongly expanded coordinate scales. Measuring frequencies: □, 1.33 kHz; ○, 234 Hz; ×, 104 Hz; △, 2.6 Hz. From Mulder *et al.* (1981).

Thermodynamics of disordered and frustrated systems – spin glasses

Assumptions and basic properties of spin glass models (MFT)

- Ergodic hypothesis – self averaging of thermodynamic potentials (free energy in thermodynamic limit equals the averaged one)
- Low-temperature phase – local magnetic moments without homogeneous magnetic order
- Degenerate thermodynamic state – ergodicity broken
- No symmetry of the Hamiltonian broken

How to reach thermodynamic limit (infinite volume)?
How to restore ergodicity? What are the order parameters?

Thermodynamics of disordered and frustrated systems – spin glasses

Assumptions and basic properties of spin glass models (MFT)

- Ergodic hypothesis – self averaging of thermodynamic potentials (free energy in thermodynamic limit equals the averaged one)
- Low-temperature phase – local magnetic moments without homogeneous magnetic order
- Degenerate thermodynamic state – ergodicity broken
- No symmetry of the Hamiltonian broken

How to reach thermodynamic limit (infinite volume)?
How to restore ergodicity? What are the order parameters?

Sherrington-Kirkpatrick mean-field solution

- Averaged free energy – replica trick with ergodic assumption (apart from critical point)

$$\beta F = -\langle \ln Z \rangle_{av} = -\lim_{\nu\to 0}\left[\frac{1}{\nu}\lim_{N\to\infty}\left(\langle Z_N^\nu\rangle_{av} - 1\right)\right] .$$

- Single order parameter $q = N^{-1}\sum_i m_i^2$
- Mean-field replica-symmetric solution: free-energy density

$$f(T,q) = -\frac{\beta}{4}(1-q)^2 - \frac{1}{\beta}\int_{-\infty}^{\infty}\mathcal{D}\eta \ln 2\cosh\left[\beta\left(h+\eta\sqrt{q}\right)\right]$$

- Global parameter: $q = N^{-1}\sum_i m_i^2 = \left\langle \tanh^2\left[\beta\left(h+\eta\sqrt{q}\right)\right]\right\rangle_\eta$ maximizes free energy!
- Inconsistency:
  - Zero temperature entropy negative: $S(0) = -\sqrt{\frac{2}{\pi}}\, k_B \approx -0.798 k_B$
  - Instability in the low-temperature phase:

$$\Lambda = 1 - \beta^2\left\langle\left(1-\tanh^2\left[\beta\left(h+\eta\sqrt{q}\right)\right]\right)^2\right\rangle_\eta < 0$$

Sherrington-Kirkpatrick mean-field solution

- Averaged free energy – replica trick with ergodic assumption (apart from critical point)

$$\beta F = -\langle \ln Z \rangle_{av} = -\lim_{\nu\to 0}\left[\frac{1}{\nu}\lim_{N\to\infty}\left(\langle Z_N^\nu\rangle_{av} - 1\right)\right] .$$

- Single order parameter $q = N^{-1}\sum_i m_i^2$
- Mean-field replica-symmetric solution: free-energy density

$$f(T,q) = -\frac{\beta}{4}(1-q)^2 - \frac{1}{\beta}\int_{-\infty}^{\infty}\mathcal{D}\eta \ln 2\cosh\left[\beta\left(h+\eta\sqrt{q}\right)\right]$$

- Global parameter: $q = N^{-1}\sum_i m_i^2 = \left\langle \tanh^2\left[\beta\left(h+\eta\sqrt{q}\right)\right]\right\rangle_\eta$ maximizes free energy!
- Inconsistency:
  - Zero temperature entropy negative: $S(0) = -\sqrt{\frac{2}{\pi}}\, k_B \approx -0.798 k_B$
  - Instability in the low-temperature phase: $\Lambda = 1-\beta^2\left\langle\left(1-\tanh^2\left[\beta\left(h+\eta\sqrt{q}\right)\right]\right)^2\right\rangle_\eta < 0$

1 Introduction – spin models and mean-field solution
- Models of interacting spins
- Models with disorder and frustration – spin glasses

2 **Fundamental concepts: Ergodicity, thermodynamic homogeneity and real replicas**
- **Ergodicity in statistical physics**
- **Real-replica method for restoring thermodynamic homogeneity**

3 Hierarchical construction of mean-field theory of spin glasses
- Discrete replica-symmetry (replica-independence) breaking
- Continuous replica-symmetry breaking

4 Solvable cases: 1RSB and asymptotic $T \nearrow T_c$ solutions
- One-level RSB – Ising
- Infinite RSB - asymptotic solution – Ising
- Potts and p-spin glass

5 Conclusions

Ergodicity in equilibrium statistical physics

- Fundamental ergodic theorem (Birkhoff)

$$\langle f \rangle_T \equiv \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} f(X(t))dt = \frac{1}{\Sigma_E}\int_{S_E} f(X)dS_E \equiv \langle f \rangle_S$$

- Phase space homogeneously covered by the phase trajectory

$$\lim_{T\to\infty} \frac{\tau_R}{T} = \frac{\Sigma_R(E)}{\Sigma(E)}$$

- Equilibrium ergodic macroscopic state
  - homogeneously spread over the allowed phase space
  - characterized by homogeneous parameters ($\{E, T\}$, $\{N, \mu\}$,...)
  - number of relevant parameters (Legendre pairs) á priori unknown

How do we determine the phase space covered by the phase space trajectory?

Homogeneity of thermodynamic potentials

- Homogeneity in the phase space

$$S(E) = k_B \ln \Gamma(E) = \frac{k_B}{\nu} \ln \Gamma(E)^{\nu} = \frac{k_B}{\nu} \ln \Gamma(\nu E)$$

$$F(T) = -\frac{k_B T}{\nu} \ln \left[\mathrm{Tr}\, e^{-\beta H}\right]^{\nu} = -\frac{k_B T}{\nu} \ln \left[\mathrm{Tr}\, e^{-\beta \nu H}\right]$$

- Homogeneity of thermodynamic potentials (Euler)

$$\alpha\, F(T, V, N, \ldots, X_i, \ldots) = F(T, \alpha V, \alpha N, \ldots, \alpha X_i, \ldots)$$

Density of the free energy $f = F/N$
– function of only densities of extensive variables X_i/N

Ergodicity (homogeneity) guarantees existence and uniqueness of the thermodynamic limit $N \to \infty$

Homogeneity of thermodynamic potentials

- Homogeneity in the phase space

$$S(E) = k_B \ln \Gamma(E) = \frac{k_B}{\nu} \ln \Gamma(E)^\nu = \frac{k_B}{\nu} \ln \Gamma(\nu E)$$

$$F(T) = -\frac{k_B T}{\nu} \ln \left[\mathrm{Tr}\, e^{-\beta H}\right]^\nu = -\frac{k_B T}{\nu} \ln \left[\mathrm{Tr}\, e^{-\beta \nu H}\right]$$

- Homogeneity of thermodynamic potentials (Euler)

$$\alpha\, F(T, V, N, \ldots, X_i, \ldots) = F(T, \alpha V, \alpha N, \ldots, \alpha X_i, \ldots)$$

Density of the free energy $f = F/N$
– function of only densities of extensive variables X_i/N

Ergodicity (homogeneity) guarantees existence and uniqueness of the thermodynamic limit $N \to \infty$

Ergodicity breaking

- Ergodicity gives meaning to statistical averages
- Thermodynamic properties in the infinite-volume limit
- Ergodicity breaking – improper statistical phase space
  1. caused by a phase transition breaking a symmetry of the Hamiltonian
  2. without apparent symmetry breaking – glass-like behavior
- Means to restore ergodicity
  1. Measurable (physical) symmetry breaking fields
  2. Real replicas (non-measurable symmetry breaking fields)

Ergodicity must be restored to establish stable equilibrium

Ergodicity breaking

- Ergodicity gives meaning to statistical averages
- Thermodynamic properties in the infinite-volume limit
- Ergodicity breaking – improper statistical phase space
  1. caused by a phase transition breaking a symmetry of the Hamiltonian
  2. without apparent symmetry breaking – glass-like behavior
- Means to restore ergodicity
  1. Measurable (physical) symmetry breaking fields
  2. Real replicas (non-measurable symmetry breaking fields)

Ergodicity must be restored to establish stable equilibrium

Real replicas – stability w.r.t. phase-space scalings

Real replicas – means to probe thermodynamic homogeneity

Replicated Hamiltonian: $[H]_\nu = \sum_{a=1}^{\nu} H^a = \sum_{\alpha=1}^{\nu} \sum_{<ij>} J_{ij} S_i^a S_j^a$

Symmetry-breaking fields: $\Delta H(\mu) = \frac{1}{2} \sum_{a \neq b} \sum_i \mu^{ab} S_i^a S_i^b$

Averaged replicated free energy with coupled replicas

$$F_\nu(\mu) = -k_B T \frac{1}{\nu} \left\langle \ln \mathrm{Tr} \exp \left\{ -\beta \sum_{a}^{\nu} H^a - \beta \Delta H(\mu) \right\} \right\rangle_{av}$$

Analytic continuation to non-integer parameter ν

Stability w.r.t. phase space scaling: $\lim_{\mu \to 0} \frac{dF_\nu(\mu)}{d\nu} \equiv 0$

Real replicas - simulate impact of surrounding bath

Real replicas – stability w.r.t. phase-space scalings

Real replicas – means to probe thermodynamic homogeneity

Replicated Hamiltonian: $[H]_\nu = \sum_{a=1}^{\nu} H^a = \sum_{\alpha=1}^{\nu} \sum_{<ij>} J_{ij} S_i^a S_j^a$

Symmetry-breaking fields: $\Delta H(\mu) = \frac{1}{2} \sum_{a \neq b} \sum_i \mu^{ab} S_i^a S_i^b$

Averaged replicated free energy with coupled replicas

$$F_\nu(\mu) = -k_B T \frac{1}{\nu} \left\langle \ln \mathrm{Tr} \exp \left\{ -\beta \sum_{a}^{\nu} H^a - \beta \Delta H(\mu) \right\} \right\rangle_{av}$$

Analytic continuation to non-integer parameter ν

Stability w.r.t. phase space scaling: $\lim_{\mu \to 0} \frac{dF_\nu(\mu)}{d\nu} \equiv 0$

Real replicas - simulate impact of surrounding bath

Annealed vs. quenched disorder

- Averaged (ν-times replicated) partition function

$$\langle Z_N^\nu\rangle_{av} = \int D[J]\mu[J] \prod_{a=1}^{\nu}\prod_{i=1}^{N} d[\mathbf{S}_i^a]\rho[\mathbf{S}_i^a] \exp\left\{-\beta\sum_{a=1}^{\nu} H[J,\mathbf{S}^a]\right\}$$

- Averaged (ν-times replicated) free energy

$$-\beta\left\langle F_N^\nu\right\rangle_{av} = \int D[J]\mu[J] \ln\int \prod_{a=1}^{\nu}\prod_{i=1}^{N} d[\mathbf{S}_i^a]\rho[\mathbf{S}_i^a] \exp\left\{-\beta\sum_{a=1}^{\nu} H[J,\mathbf{S}^a]\right\}$$

- Replicas for disordered systems:
  - Quenched disorder (spin glasses) – replica trick ($\nu \to 0$)

$$\beta F_{qu} = -\lim_{\nu\to 0}\left[\frac{1}{\nu}\lim_{N\to\infty}\left(\langle Z_N^\nu\rangle_{av} - 1\right)\right]$$

  - Annealed disorder – thermodynamic homogeneity (ν arbitrary)

$$\beta F_{an} = -\frac{1}{\nu}\lim_{N\to\infty}\ln\langle Z_N^\nu\rangle_{av}$$

Ergodicity breaking – broken LRT in replicated space

- Breaking of LRT to inter-replica interaction $\mu^{ab} \to 0$

$$f_\nu = \frac{\beta J^2}{4}\left[\frac{1}{\nu}\sum_{a\neq b}^{\nu}\left\{\left(\chi^{ab}\right)^2 + 2q\chi^{ab}\right\} - (1-q)^2\right]$$

$$-\frac{1}{\beta\nu}\int_{-\infty}^{\infty}\frac{d\eta}{\sqrt{2\pi}}\,e^{-\eta^2/2}\ln \mathrm{Tr}_\nu \exp\left\{\beta^2 J^2\sum_{a<b}^{\nu}\chi^{ab}S^aS^b + \beta\bar{h}\sum_{a=1}^{\nu}S^a\right\}$$

$$\chi^{ab} = \langle\langle S^aS^b\rangle_T\rangle_{av} - q\,,\; q = \langle\langle S^a\rangle_T^2\rangle_{av}\,,\; \bar{h} = h + \eta\sqrt{q}$$

- Free energy f_ν must be analytic function of index ν
- Parisi conditions for analytic continuation

$$\chi^{aa} = 0\,,\quad \chi^{ab} = \chi^{ba}\,,\quad \sum_{c=1}^{\nu}\left(\chi^{ac} - \chi^{bc}\right) = 0$$

- $K < \nu - 1$ different inter-replica susceptibilities $\chi_1, \ldots, \chi_K$ with multiplicities $\nu_1, \ldots, \nu_K$

Ergodicity breaking – broken LRT in replicated space

- Breaking of LRT to inter-replica interaction $\mu^{ab} \to 0$

$$f_\nu = \frac{\beta J^2}{4}\left[\frac{1}{\nu}\sum_{a\neq b}^{\nu}\left\{\left(\chi^{ab}\right)^2 + 2q\chi^{ab}\right\} - (1-q)^2\right]$$

$$-\frac{1}{\beta\nu}\int_{-\infty}^{\infty}\frac{d\eta}{\sqrt{2\pi}}\, e^{-\eta^2/2}\ln \mathrm{Tr}_\nu \exp\left\{\beta^2 J^2 \sum_{a<b}^{\nu}\chi^{ab}S^aS^b + \beta\bar{h}\sum_{a=1}^{\nu}S^a\right\}$$

$$\chi^{ab} = \langle\langle S^aS^b\rangle_T\rangle_{av} - q\,,\ q = \langle\langle S^a\rangle_T^2\rangle_{av}\,,\ \bar{h} = h + \eta\sqrt{q}$$

- Free energy f_ν must be analytic function of index ν
- Parisi conditions for analytic continuation

$$\chi^{aa} = 0\,,\quad \chi^{ab} = \chi^{ba}\,,\quad \sum_{c=1}^{\nu}\left(\chi^{ac} - \chi^{bc}\right) = 0$$

- $K < \nu - 1$ different inter-replica susceptibilities $\chi_1, \ldots, \chi_K$ with multiplicities $\nu_1, \ldots, \nu_K$

Analytic continuation

- Only specific matrices $\nu \times \nu$ allow for analytic continuation to real ν
- Multiplicity of the order parameters – K different values

$$\begin{pmatrix} 0 & q_0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_0 & 0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & 0 & q_0 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & q_0 & 0 & q_2 & q_2 & q_2 & q_2 \\ q_2 & q_2 & q_2 & q_2 & 0 & q_0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_0 & 0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & 0 & q_0 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & q_0 & 0 \end{pmatrix}$$

$q_l = q + \chi_l\,,\ \nu_l = 2^l\,,\ \nu - 1 = \sum_l^K \nu_l$

Analytic continuation

- Only specific matrices $\nu \times \nu$ allow for analytic continuation to real ν
- Multiplicity of the order parameters – K different values

$$\begin{pmatrix} 0 & q_0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_0 & 0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & 0 & q_0 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & q_0 & 0 & q_2 & q_2 & q_2 & q_2 \\ q_2 & q_2 & q_2 & q_2 & 0 & q_0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_0 & 0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & 0 & q_0 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & q_0 & 0 \end{pmatrix}$$

$q_l = q + \chi_l$, $\nu_l = 2^l$, $\nu - 1 = \sum_l^K \nu_l$

Analytic continuation

- Only specific matrices $\nu \times \nu$ allow for analytic continuation to real ν
- Multiplicity of the order parameters – K different values

$$\begin{pmatrix} 0 & q_0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_0 & 0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & 0 & q_0 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & q_0 & 0 & q_2 & q_2 & q_2 & q_2 \\ q_2 & q_2 & q_2 & q_2 & 0 & q_0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_0 & 0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & 0 & q_0 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & q_0 & 0 \end{pmatrix}$$

$q_l = q + \chi_l$, $\nu_l = 2^l$, $\nu - 1 = \sum_l^K \nu_l$

Analytic continuation

- Only specific matrices $\nu \times \nu$ allow for analytic continuation to real ν
- Multiplicity of the order parameters – K different values

$$\begin{pmatrix} 0 & q_0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_0 & 0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & 0 & q_0 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & q_0 & 0 & q_2 & q_2 & q_2 & q_2 \\ q_2 & q_2 & q_2 & q_2 & 0 & q_0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_0 & 0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & 0 & q_0 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & q_0 & 0 \end{pmatrix}$$

$q_l = q + \chi_l$, $\nu_l = 2^l$, $\nu - 1 = \sum_l^K \nu_l$

Analytic continuation

- Only specific matrices $\nu \times \nu$ allow for analytic continuation to real ν
- Multiplicity of the order parameters – K different values

$$\begin{pmatrix} 0 & q_0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_0 & 0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & 0 & q_0 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & q_0 & 0 & q_2 & q_2 & q_2 & q_2 \\ q_2 & q_2 & q_2 & q_2 & 0 & q_0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_0 & 0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & 0 & q_0 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & q_0 & 0 \end{pmatrix}$$

$q_l = q + \chi_l \, , \nu_l = 2^l \, , \nu - 1 = \sum_l^K \nu_l$

Analytic continuation

- Only specific matrices $\nu \times \nu$ allow for analytic continuation to real ν
- Multiplicity of the order parameters – K different values

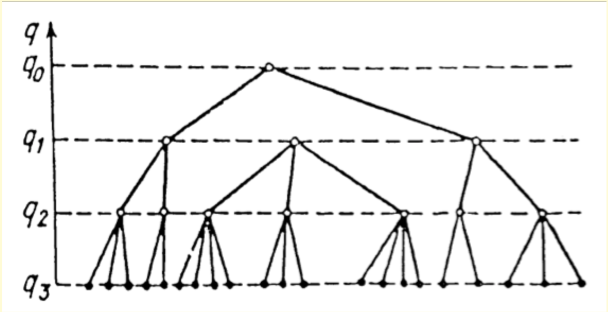
$$\begin{pmatrix} 0 & q_0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_0 & 0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & 0 & q_0 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & q_0 & 0 & q_2 & q_2 & q_2 & q_2 \\ q_2 & q_2 & q_2 & q_2 & 0 & q_0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_0 & 0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & 0 & q_0 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & q_0 & 0 \end{pmatrix}$$

$q_l = q + \chi_l \, , \nu_l = 2^l \, , \nu - 1 = \sum_l^K \nu_l$

Ultrametric structure

Ultrametric structure

- only block matrices of identical elements
- larger blocks multiples of smaller blocks
- hierarchy of embeddings around diagonal
- ultrametric metrics (tree-like)



Multiple replica hierarchies

Averaged free energy density with K hierarchies of replicas

$\Delta\chi_l = \chi_l - \chi_{l+1} \geq \Delta\chi_{l+1} \geq 0$, ν_l – arbitrary positive

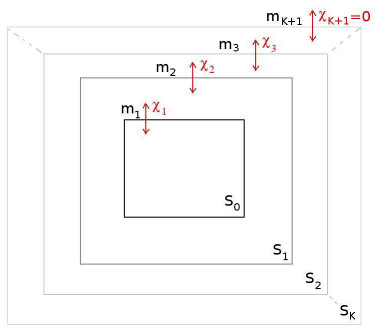
$$f_K(q; \Delta\chi_1, \ldots, \Delta\chi_K, \nu_1, \ldots, \nu_K) = -\frac{\beta}{4}\left(1 - q - \sum_{l=1}^{K}\Delta\chi_l\right)^2 - \frac{1}{\beta}\ln 2$$
$$+ \frac{\beta}{4}\sum_{l=1}^{K}\nu_l\Delta\chi_l\left[2\left(q + \sum_{i=l}^{K}\Delta\chi_i\right) - \Delta\chi_l\right] - \frac{1}{\beta}\int_{-\infty}^{\infty}\mathcal{D}\eta\ \ln\ Z_K$$

Hierarchical local partition sums $Z_l = \left[\int_{-\infty}^{\infty}\mathcal{D}\lambda_l\, Z_{l-1}^{\nu_l}\right]^{1/\nu_l}$

Initial condition $Z_0 = \cosh\left[\beta\left(h + \eta\sqrt{q} + \sum_{l=1}^{K}\lambda_l\sqrt{\Delta\chi_l}\right)\right]$

Gaussian measure $\mathcal{D}\lambda \equiv d\lambda\ e^{-\lambda^2/2}/\sqrt{2\pi}$

Multiple embeddings – including boundary terms



- $\Delta\chi_l$ – inter-replica interaction strength,
  λ_l – effective magnetic field due to replicated spins
- $\nu_l V$: volume affected by replicated spins – range of inter-replica interaction

$$\frac{N}{V}\ln Z_{l-1}(\beta,\bar{h}_l)$$
$$\longrightarrow \frac{N}{\nu_l V}\ln\int\mathcal{D}\lambda_l\, Z_{l-1}^{\nu_l}\left(\beta,\bar{h}_l+\lambda_l\sqrt{\Delta\chi_l}\right)$$

- Effective weight of surrounding spins in thermal averaging

$$\rho_l=\frac{Z_{l-1}^{\nu_l}}{\langle Z_{l-1}^{\nu_l}\rangle_{\lambda_l}}$$

Equilibrium state – stationarity equations & stability

- Stationarity equations with discrete K replica hierarchies

$$q = \langle\langle t\rangle_K^2\rangle_\eta \ ,$$
$$\Delta\chi_l = \langle\langle\langle t\rangle_{l-1}^2\rangle_K\rangle_\eta - \langle\langle\langle t\rangle_l^2\rangle_K\rangle_\eta \ ,$$
$$\nu_l\Delta\chi_l = \frac{4}{\beta^2}\frac{\langle\langle\ln Z_{l-1}\rangle_K\rangle_\eta - \langle\langle\ln Z_l\rangle_K\rangle_\eta}{2\left(q + \sum_{i=l+1}^K \Delta\chi_i\right) + \Delta\chi_l}$$

$$t \equiv \tanh\left[\beta\left(h + \eta\sqrt{q} + \sum_{l=1}^K \lambda_l\sqrt{\Delta\chi_l}\right)\right] ,$$
$$\langle t\rangle_l(\eta;\lambda_K,\ldots,\lambda_{l+1}) = \langle\rho_l\ldots\langle\rho_1 t\rangle_{\lambda_1}\ldots\rangle_{\lambda_l} , \ \rho_l = Z_{l-1}^{\nu_l}/\langle Z_{l-1}^{\nu_l}\rangle_{\lambda_l}$$

- $K+1$ stability conditions determine number K

$$\Lambda_l^K = 1 - \beta^2\left\langle\left\langle\left\langle 1 - t^2 + \sum_{i=0}^{l}\nu_i\left(\langle t\rangle_{i-1}^2 - \langle t\rangle_i^2\right)\right\rangle_l^2\right\rangle_K\right\rangle_\eta \geq 0$$

Infinite many replica hierarchies I

Limit to infinite number of replica hierarchies $K \to \infty$

- Infinitesimal differences $\Delta\chi_l$ and $\Delta\nu_l$:
  $\Delta\chi_l = \Delta\chi/K$, $\Delta\nu_l = \Delta m/K$, $\Delta\chi_l/\Delta\nu_l \to x(m) < \infty$
- Parisi continuous free energy (around 1RSB):

$$f(q,\chi_1,m_1,m_0;x(m)) = -\frac{\beta}{4}(1-q-\chi_1-X_0(m_1))^2+\frac{\beta}{4}\Big[m_1\left(q+\chi_1+X_0(m_1)\right)^2$$

$$-m_0 q^2\Big] - \frac{\beta}{4}\int_{m_0}^{m_1} dm\,[q+\chi_1+X_0(m)]^2 - \frac{1}{\beta}\left\langle g_1(m_0,h+\eta\sqrt{q})\right\rangle_\eta$$

- Integral representation of the interacting part

$$g_1(m_0,h) = \mathbb{E}_0(m_0,m_1,h)\circ g_1(h)$$

$$\equiv \overline{\mathbb{T}}_m \exp\left\{\frac{1}{2}\int_{m_0}^{m_1} dm\, x(m)\left[\partial^2_{\bar h} + m g_1'(m;h+\bar h)\partial_{\bar h}\right]\right\} g_1(h+\bar h)\Big|_{\bar h=0}$$

$$g_1'(m;h) = \partial g_1(m,h)/\partial h$$

Infinite many replica hierarchies II

- Anti time-ordering product from quantum many-body PT

$$\overline{\mathbb{T}}_\lambda \exp\left\{\int_0^1 d\lambda \widehat{O}(\lambda)\right\} \equiv 1+\sum_{n=1}^{\infty}\int_0^1 d\lambda_1 \int_0^{\lambda_1} \ldots \int_0^{\lambda_{n-1}} d\lambda_n \widehat{O}(\lambda_n)\ldots\widehat{O}(\lambda_1)$$

- Initial condition (1RSB)

$$g_1(h) \equiv g_1(m_1, h) = \frac{1}{m_1}\ln\int_{-\infty}^{\infty}\frac{d\phi}{\sqrt{2\pi}}e^{-\phi^2/2}\left[2\cosh\left(\beta(h+\phi\sqrt{\chi_1})\right)\right]^{m_1}$$

- Closed implicit equation

$$g_1'(m,h) = \mathbb{E}(m,m_1,h)\circ g_0(h)$$
$$\equiv \overline{\mathbb{T}}_m \exp\left\{\frac{1}{2}\int_m^{m_1} dm'\ x(m')\left[\partial_{\bar{h}}^2 + 2m' g_1'(m';h+\bar{h})\partial_{\bar{h}}\right]\right\} g_1'(h+\bar{h})\Big|_{\bar{h}=0}$$

- Notation: $X_0(m) = \int_{m_0}^{m} dm' x(m')$

Discrete vs. continuous replica-symmetry breaking

Discrete RSB

1. Hierarchical embeddings – ultrametric structure
2. No restriction on the replica-induced order parameters
3. The number of replica hierarchies K from stability conditions
4. Either unstable or locally stable

Continuous RSB

1. Limit of infinite number of replica hierarchies
2. Infinitesimal distance between replica hierarchies
3. Closed theory independently of stability of the discrete scheme
4. Always marginally stable

1 Introduction - spin models and mean-field solution
- Models of interacting spins
- Models with disorder and frustration - spin glasses

2 Fundamental concepts: Ergodicity, thermodynamic homogeneity and real replicas
- Ergodicity in statistical physics
- Real-replica method for restoring thermodynamic homogeneity

3 Hierarchical construction of mean-field theory of spin glasses
- Discrete replica-symmetry (replica-independence) breaking
- Continuous replica-symmetry breaking

4 Solvable cases: 1RSB and asymptotic $T \nearrow T_c$ solutions
- One-level RSB – Ising
- Infinite RSB - asymptotic solution – Ising
- Potts and p-spin glass

5 Conclusions

First level replica-symmetry (ergodicity) breaking

- Ergodicity broken in the SG phase – one embedding

$$f(q;\chi,\nu) = -\frac{\beta}{4}(1-q)^2 + \frac{\beta}{4}(\nu-1)\chi(2q+\chi) + \frac{\beta}{2}\chi$$
$$- \frac{1}{\beta\nu}\int_{-\infty}^{\infty}\mathcal{D}\eta \ln \int_{-\infty}^{\infty}\mathcal{D}\lambda \left\{2\cosh\left[\beta\left(h+\eta\sqrt{q}+\lambda\sqrt{\chi}\right)\right]\right\}^{\nu}$$

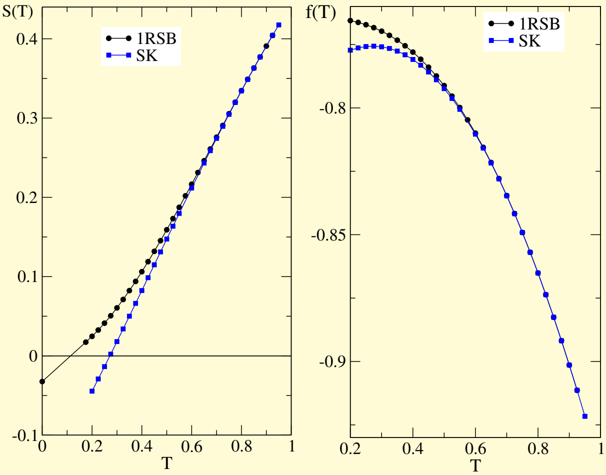
- Stationarity equations $\left(t \equiv \tanh[\beta(h+\eta\sqrt{q}+\lambda\sqrt{\chi})]\right)$

$$q = \langle\langle t\rangle_\lambda^2\rangle_\eta \ , \qquad q_{EA} = q+\chi = \langle\langle t^2\rangle_\lambda\rangle_\eta$$
$$\beta^2\chi(2q+\chi)\nu = \Big[\langle \ln\cosh[\beta(h+\eta\sqrt{q}+\lambda\sqrt{\chi})]\rangle_\lambda$$
$$- \ln\left\langle \cosh^{\nu}[\beta(h+\eta\sqrt{q}+\lambda\sqrt{\chi})]\right\rangle_\lambda^{1/\nu}\Big]$$

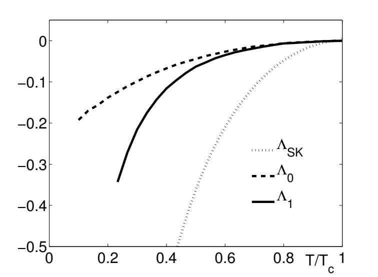
- Stability conditions

$$\Lambda_0 = 1 - \beta^2\left\langle\left\langle(1-t)^2\right\rangle_\lambda\right\rangle_\eta$$
$$\Lambda_1 = 1 - \beta^2\left\langle\left\langle 1-(1-\nu)t^2 - \nu\left\langle t^2\right\rangle_\lambda\right\rangle_\lambda^2\right\rangle_\eta$$

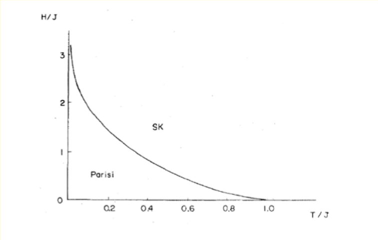
1RSB - thermodynamics



RS & 1RSB - instability



Instability at zero magnetic field



SG phase in magnetic field - AT line

SK model at zero magnetic field

Only asymptotic expansions available for $K \to \infty$

- Small expansion parameter $\tau = (T_c - T)/T_c$

$$\Delta\chi_l^K \doteq \frac{2}{2K+1}\,\tau\,, \qquad \nu_l^K \doteq \frac{4(K-l+1)}{2K+1}\,\tau, \qquad q^K \doteq \frac{1}{2K+1}\,\tau\,,$$

$$Q^K \equiv q_{EA} = q + \chi_1 - \chi_K \doteq \tau + \frac{12K(K+1)+1}{3(2K+1)^2}\,\tau^2\,, \qquad \Lambda_l^K \doteq -\frac{4}{3}\,\frac{\tau^2}{(2K+1)^2}$$

$$\chi_T \doteq \beta\left(1 - Q^K + \sum_{l=1}^{K} m_l \Delta\chi_l\right) \doteq 1 - \frac{\tau^2}{3(2K+1)^2}$$

$$\Delta f \doteq \left(\frac{1}{6}\tau^3 + \frac{7}{24}\tau^4 + \frac{29}{120}\tau^5\right) - \frac{1}{360}\tau^5\left(\frac{1}{K}\right)^4$$

Parisi continuous ansatz proven right

SK model in magnetic field

- Full RSB at AT line reduces to 1RSB ($h > 0$)
- Small expansion parameter $\alpha = \beta^2 \left\langle (1 - t_0^2)^2 \right\rangle_\eta - 1$
  ($t_0 \equiv \tanh[\beta(h + \eta\sqrt{q})]$)

$$\nu = \frac{2\langle t_0^2 (1 - t_0^2)^2 \rangle_\eta}{\langle (1 - t_0^2)^3 \rangle_\eta}$$

$$\chi_1 = \frac{1}{2\beta^2 \nu} \, \frac{\beta^2 \left\langle (1 - t_0^2)^2 \right\rangle_\eta - 1}{1 - 3\beta^2 \left\langle t_0^2 (1 - t_0^2)^2 \right\rangle_\eta} + O(\alpha^2)$$

$$\nu_l^K \doteq \nu_1 + (K + 1 - 2l)\Delta\nu/K, \qquad \Delta\chi_l^K \doteq \chi_1/K$$

$$\Delta\nu \doteq \frac{\beta^2 \chi_1 \left\langle (1 - t_0^2)^2 \left(2\left(1 - 3t_0^2\right)^2 + 3\left(t_0^2 - 1\right)\nu\left(8t_0^2 + \left(t_0^2 - 1\right)\nu\right)\right)\right\rangle_\eta}{\left\langle (1 - t_0^2)^3 \right\rangle_\eta}$$

$$\Lambda_l^K \doteq -\frac{2\beta^2}{3K^2} \, \frac{\chi_1 \Delta\nu}{\nu + 2}$$

Potts glass ($p < 4$): discrete RSB

- Two 1RSB solutions for $\nu_1 \doteq \frac{p-2}{2} + \frac{36-12p+p^2}{8(4-p)}\tau$
  - Locally stable solution (near T_c and $p > p^* \approx 2.82$)

$$q^{(1)} \doteq 0\,, \quad \Delta\chi^{(1)} \doteq \frac{2}{4-p}\tau$$

Stability function: $\Lambda_1^{(1)} \doteq \frac{\tau^2(p-1)}{6(4-p)^2}\left(7p^2 - 24p + 12\right)$

  - Unstable solution ($p > p^*$ unphysical)

$$q^{(2)} \doteq \frac{-12 + 24p - 7p^2}{3(4-p)^2(p-2)}\tau^2\,, \quad \Delta\chi^{(2)} \doteq \frac{2}{4-p}\tau$$

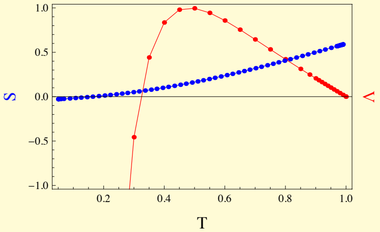
- K RSB (from the unstable one)

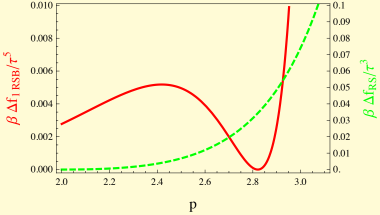
$$q^K \doteq -\frac{1}{3K^2}\frac{12 - 24p + 7p^2}{(4-p)^2(p-2)}\tau^2\,, \quad \Delta\chi_l^K \doteq \frac{1}{K}\frac{2}{(4-p)}\tau\,,$$

$$\nu_l^K \doteq \frac{p-2}{2} + \frac{2}{4-p}\left[3 + \frac{3}{2}p - p^2 + \left(3 - 6p + \frac{7}{4}p^2\right)\frac{2l-1}{2K}\right]\tau$$

Potts glass ($p = 3$): coexistence

1RSB a FRSB coexist near T_c





Stability and entropy of 1RSB solution ($p = 3$)

Free-energy difference as function of p

Free-energy differences:

$$\beta(f_c - f_{1RSB}) \doteq \frac{(p-1)(p(7p-24)+12)^2\tau^5}{720(4-p)^5}\ ,\quad \beta(f_c - f_{RS}) \doteq \frac{(p-1)(p-2)^2\tau^3}{3(4-p)(6-p)^2}$$

p-spin glass: 1RSB I

- Discontinuous transition to the low-temperature phase for $p > 2$
- Asymptotic solution $p \to \infty$: 1RSB

$$
\begin{aligned}
& f_T^{(p\to\infty)}(q, \chi_1, \mu_1) \\
& \quad = -\frac{1}{4T}\left[1 - (q+\chi_1)\left(1 - \ln\left(q+\chi_1\right)\right] - \frac{1}{\mu_1}\ln\left[2\cosh(\mu_1 h)\right]\right. \\
& -\frac{\mu_1}{4}\left[\chi_1 - (q+\chi_1)\ln\left(q+\chi_1\right)\right] - \frac{\mu_1 q}{4}\left[\ln q + p\left(1 - \tanh^2(\mu_1 h)\right)\right]
\end{aligned}
$$

rescaled variable $\mu_1 = \beta\nu_1$

- Low-temperature solution ($p = \infty$) – Random energy model

$$
\chi_1 = 1 - q\,, \qquad q = \exp\{-p(1 - \tanh^2(\mu_1 h))\}\,,
$$

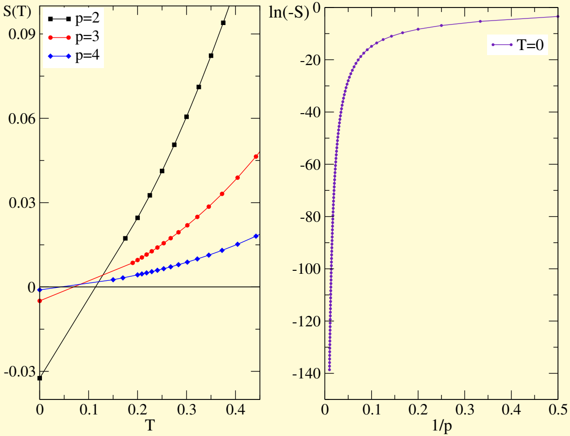
$$
\mu_1 = 2\sqrt{\ln\left[2\cosh(\mu_1 h)\right] - h\tanh(\mu_1 h)}
$$

for $\beta > 2\sqrt{\ln\left[2\cosh(\beta h)\right] - h\tanh(\beta h)}$,

otherwise $q + \chi_1 = 0$ and $\mu_1 = \beta$

p-spin glass: 1RSB II

- Negative entropy for $p < \infty$
  – full continuous free energy around 1RSB needed



Conclusions

Spin-glass phase: Ergodicity breaking without symmetry breaking

1. Frustration with disorder prevents existence of physical symmetry-breaking fields
2. Real replicas – means to test thermodynamic homogeneity (ergodicity)
3. Analytic continuation to non-integer replication index mandatory – ultrametric structure
4. Broken LRT of inter-replica interaction – broken replica symmetry (ergodicity)
5. Hierarchical replications – series of admissible solutions (equilibrium states)
6. Local and global stability conditions select the true equilibrium
7. Continuous RSB – marginally stable (available only via asymptotic expansions)